Homework Assignment II

Physics 105.2, Instructor: Petr Hořava

This assingment is due Friday, Feb 14.

The central concept of this week's lectures was that of a Lagrangian, $L = T - V$, leading to the reformulation of Newtonian dynamics in terms of the Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0,$$

with $i = 1, \ldots N$ where N is the number of DoF of the system. The switch from the Newtonian formulation of mechanics (which is familiar to you from the freshman mechanics course) to the Lagrangian formulation is perhaps the most radical and difficult paradigm shift that you will encounter in this entire course. Thus, getting used to the Lagrangian, and developing sufficient intuition for this concept, takes some time and effort. This homework assignment is aimed at developing some of that intuition, and is therefore naturally more demanding (but also more exciting :-) that last week's. Enjoy!

First, some required reading: Sections 1.1-1.9 and Appendix B of Chapter 1 of [Hand-Finch].

1. [Problem 1-6(a) of Hand-Finch] (Physically equivalent Lagrangians)
   Prove that adding a constant to the Lagrangian L or else multiplying the Lagrangian by a constant produces a new Lagrangian L' that is physically equivalent to L. What we mean by physically equivalent is that the Lagrange equations (as written above) for $q_i(t)$ are equivalent under this change of the Lagrangian.

2. [Problem 1-6(b) of Hand-Finch] (Physically equivalent Lagrangians – cont.)
   There is even more freedom to change the Lagrangian *without changing the physics it describes.* A total time derivative of an arbitrary function of the dynamical variables can be added to the Lagrangian to produce a completely equivalent Lagrangian. Consider a new Lagranginan L' which is produced as follows:

   $$L' = L + \frac{dF}{dt}.$$

   We assume that F is an arbitrary function of t and q_i but not of the generalized velocities $\dot{q}_i$. Prove that the Lagrange equations for $q(t)$ are invariant under this change of the Lagrangian. Thus, since one can always make transformations of this sort, the Lagrangian for a given physical system is not unique.

3. [part of Problem 1-7 of [HF]] (Guessing the Lagrangian for a free particle)
   Assume that you don't know about kinetic energy or Newton's Laws of motion. Suppose instead of deriving the Lagrange equations we postulated them. We define the basic law of mechanics to be these equations, and ask ourselves the question: What is

the Lagrangian for a free particle? (This is a particle in an empty three-dimensional space with no forces acting on it. Be sure to set up an inertial reference system – an inertial frame.)

(a) Explain why, on very general grounds, L cannot be a function of the Cartesian coordinates x, y, z. It also cannot depend on the individual components of the velocity vector, in any way except as a function of the magnitude (squared) of the velocity: $v^2 \equiv v_x^2 + v_y^2 + v_z^2$. On what assumption about the properties of space does this depend?

(b) The simplest choice might be to guess it must be proportional to v^2, the magnitude (squared) of the velocity in an inertial frame (which we will call K). Take $L = v^2$. A second inertial frame (call it K') moves at the constant velocity $-\vec{V}_0$ with respect to K, so that the transformation law of velocities is

$$\vec{v}' = \vec{v} + \vec{V}_0.$$

Prove that $L' \equiv (v')^2$ is a possible choice for the Lagrangian in the frame K' (i.e., prove that the corresponding Lagrange equations are equivalent to those of the Lagrangian $L = v^2$ describing the system in the original frame K). Explain how this proves that all inertial frames are equivalent. You will have to make use of the result of the previous problem to show this. With this approach we prove the equivalence of inertial frames from the form of the Lagrangian (!), instead of postulating this equivalence at the start, which is the usual way of doing things.

4. [Problem 1-12 of [HF]] (L for free particle in plane polar coordinates)
Express the Lagrangian for a free particle moving in a plane in plane-polar coordinates r, θ. From this prove that, in terms of radial and tangential components, the acceleration in polar coordinates is

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{e}}_\theta,$$

where $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$ are unit vectors in the positive radial and tangential directions.

5. (Transformation properties of Lagrange's equations under a change of coordinates)
Consider a dynamical system described by Lagrangian L which is a function of time t, N generalized coordinates q_i, and N generalized velocities $\dot{q}_i$, with index i going from 1 to N. Introduce a new coordinate system $\tilde{t}$, $\tilde{q}_i$ and $\dot{\tilde{q}}_i$, related to the original coordinates by the following coordinate transformation:

$$\begin{aligned}
\tilde{q}_j &= \tilde{q}_j(q_i, t), \\
\dot{\tilde{q}}_j &= \sum_k \frac{\partial \tilde{q}_j}{\partial q_k}\dot{q}_k + \frac{\partial \tilde{q}_j}{\partial t}, \\
\tilde{t} &= t.
\end{aligned}$$

Prove that under this change of coordinates, the left-hand-side of the Lagrange equations transforms nicely as a vector, i.e.,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_j \frac{\partial \tilde{q}_j}{\partial q_i}\left[\frac{d}{d\tilde{t}}\left(\frac{\partial L}{\partial \dot{\tilde{q}}_j}\right) - \frac{\partial L}{\partial \tilde{q}_j}\right]$$

if we assume that L transforms as a scalar; i.e.,

$$L(\tilde{t}, \tilde{q}_i, \dot{\tilde{q}}_i) = L(\tilde{t}(t), \tilde{q}_i(q_j, t), \dot{\tilde{q}}_i(t, q_j, \dot{q}_j)).$$

(If the general proof for any value of N seems difficult try the simple case of $N = 1$ first.)

Notice that while the left-hand-side of Lagrange's equations has these nice, vectorial tranformation properties under the general change of coordinates, neither of the two parts of the left-hand-side,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \quad \text{or} \quad \frac{\partial L}{\partial q_i}$$

transforms nicely as a vector.

The nice geometric transformation properties of Lagrange's equations under a very general coordinate tranformation have powerful and liberating implications: In the Lagrangian formulation of mechanics, we are now free to choose any coordinate system that we find useful, and in particular we do not have to formulate the laws of mechanics using only inertial frames.